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Kinetic equations for Bose-Einstein condensates from the 2PI effective action

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Abstract

We use the 2PI effective action of a relativistic scalar field theory to derive kinetic equations for a Bose-condensed system near the phase transition. We start from equations of motions derived within a $\frac{1}{N}$-expansion at NLO. In taking the non-relativistic limit we obtain a generalized Gross-Pitaevskii equation for the condensate field. Within the Popov approximation we explicitly compute the collision term up to order g^2 using the Kadanoff-Baym formalism. For the sake of self-consistency we derive in the same way a Boltzmann equation for the non-condensate distribution function. The final results are in agreement to those previously obtained by Griffin, Nikuni and Zaremba and by Stoof.

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1 Introduction

Since the first experimental realization of a Bose condensed dilute gas in 1995 there has been an intense effort to further understand this new state of matter both from the experimental and theoretical point of view. The most common theoretical ansatz for a wide range of questions is the famous Gross-Pitaevskii equation of motion (e.o.m.) [1]. This equation for the Bose macroscopic wavefunction Φ gives a complete description of the condensate at $T = 0$. Near the phase transition one clearly has to take into account the effects of the non-condensate atoms. One approach to this problem is to derive a useful kinetic equation that includes the collisions between the condensate and non-condensate part of the system. This was done by Griffin, Nikuni and Zaremba [2] starting from a suitably generalized Gross-Pitaevskii equation (see also the work by Imamovic-Tomasovic and Griffin [3, 4, 5]). Further approaches to a quantum kinetic description of Bose condensed systems can, e.g., be found in [6, 7, 8, 9].
In this paper we apply a method coming from non-equilibrium relativistic quantum field theory. For a recent review of the so called two particle irreducible (2PI) effective action techniques see [10]. This powerful method has already been used in a variety of interesting issues like physics of the early universe and heavy ion collisions [11, 12, 13], and recently to the non-equilibrium dynamics of Bose-Einstein condensates [14]. We use the e.o.m. for a N-component scalar field obtained within a $\frac{1}{N}$-expansion of the 2PI effective action to next to leading order (NLO) [15]. We finally arrive at the same kinetic equation previously derived in [2, 6].

2 Field equation of motion

Our starting point is the relativistic e.o.m. from the $\frac{1}{N}$-expansion of the 2PI effective action at NLO. For a N-component real scalar field ϕ_a with mass m and $\frac{\lambda}{4!}\phi^4$-interaction one has (see (27) in [15])

$$-\left(\Box + m^2 + \frac{\lambda}{6N}\left[\Phi_c(x)\Phi_c(x) + G_{cc}(x,x)\right]\right)\Phi_a(x) = K_a(x,x), \qquad (1)$$

with (see (35) in [15])

$$\begin{aligned} K_a(x,y) = &\frac{\lambda}{3N}\Phi_b(x)G_{ba}(x,y) \\ &-i\frac{\lambda}{6N}\int_{\mathcal{C}} dz^0 \int d^3z G_{bc}(x,z)G_{bc}(x,z)K_a(z,y), \end{aligned} \qquad (2)$$

and the propagator G given by

$$G_{ab}(x,y) =< T\phi_a(x)\phi_b(y) > - < \phi_a(x) >< \phi_b(y) > . \tag{3}$$

The mean field is denoted by $\Phi_a(x) =< \phi_a(x) >$. In (3) T means time ordering along the Schwinger-Keldysh contour [16, 17, 18, 19]. The z^0-integral in (2) is taken along this contour as well (see fig.1).
As we ultimately want to describe a Bose condensed gas we have to take $N = 2$, or $a = 1, 2$.
Next we perform the non-relativistic limit by the following replacements

$$\Phi_a \longrightarrow \frac{1}{\sqrt{m}}\Phi_a^{NR}, \tag{4}$$

$$G \longrightarrow \frac{1}{m}G^{NR}, \tag{5}$$

$$\Pi \longrightarrow \frac{1}{m^2}\Pi^{NR}, \tag{6}$$

$$K_a \longrightarrow 2\sqrt{m}K_a^{NR}, \tag{7}$$

$$\lambda \longrightarrow 12m^2 g, \tag{8}$$

$$(\Box + m^2) \longrightarrow -2m(i\partial_{x^0} + \frac{\triangle}{2m}), \tag{9}$$

where we define

$$\Pi(x,y) = -\frac{1}{2}G_{ab}(x,y)G_{ab}(x,y). \tag{10}$$

The dimensionful prefactors arise because of the different normalizations of relativistic and non-relativistic fields. Throughout the paper we set $\hbar = c = 1$.
The non-relativistic e.o.m. now reads

$$\left(i\partial_{x^0} + \frac{\triangle}{2m} - \frac{g}{2}\left[\Phi_c^{NR}(x)\Phi_c^{NR}(x) + G_{cc}^{NR}(x,x)\right]\right)\Phi_a^{NR}(x) =$$
$$g\Phi_b^{NR}(x)G_{ba}^{NR}(x,x) + 2ig\int_{\mathcal{C}} dz^0 \int d^3z \quad \Pi^{NR}(x,z)K_a^{NR}(z,x). \tag{11}$$

For a later comparison to other work on Bose condensed systems it is convenient to express (11) in terms of one complex field Φ. Thus we perform the following redefinition (we suppress from now on the 'NR' index)

$$\phi_1 \longrightarrow \frac{1}{\sqrt{2}}(\Psi^* + \Psi), \qquad \phi_2 \longrightarrow \frac{i}{\sqrt{2}}(\Psi^* - \Psi), \tag{12}$$

with

$$\Psi = \Phi + \tilde{\Psi}, \tag{13}$$

and correspondingly for the expectation values

$$< \Psi >= \Phi, \qquad < \tilde{\Psi} >= 0. \tag{14}$$

The interpretations of Φ as the condensate and $\tilde{\Psi}$ as non-condensate complex fields are obvious. The components of the propagator matrix G_{ab} are replaced by

$$G_{11}(x,x) \longrightarrow \frac{1}{2}(\tilde{m}^* + 2\tilde{n} + \tilde{m}), \tag{15}$$

$$G_{22}(x,x) \longrightarrow -\frac{1}{2}(\tilde{m}^* - 2\tilde{n} + \tilde{m}), \tag{16}$$

$$G_{12}(x,x) \longrightarrow \frac{i}{2}(\tilde{m}^* - \tilde{m}), \tag{17}$$

$$G_{21}(x,x) = G_{12}(x,x), \tag{18}$$

where we introduced the non-equilibrium non-condensate densities

$$\tilde{n}(x) =< \tilde{\Psi}^*(x)\tilde{\Psi}(x) >, \qquad \tilde{m}(x) =< \tilde{\Psi}(x)\tilde{\Psi}(x) > . \tag{19}$$

The e.o.m. (11) reads in terms of the new fields

$$\left(i\partial_{x^0} + \frac{\triangle}{2m} - g\left[n_c(x) + 2\tilde{n}(x)\right]\right)\Phi(x) - g\tilde{m}(x)\Phi^*(x) =$$
$$2ig\int_{\mathcal{C}} dz^0 \int d^3z \Pi(x,z)K(z,x), \tag{20}$$

with

$$\begin{aligned} K(x,y) &= \frac{1}{\sqrt{2}}(K_1(x,y) + iK_2(x,y)) \\ &= g\Phi(x) < T\tilde{\Psi}^*(x)\tilde{\Psi}(y) > + g\Phi^*(x) < T\tilde{\Psi}(x)\tilde{\Psi}(y) > \\ &+ 2ig\int_{\mathcal{C}} dz^0 \int d^3z \Pi(x,z)K(z,y), \end{aligned} \tag{21}$$

and the non-equilibrium density in the condensate $n_c(x) = \frac{1}{2}(\Phi_1^2(x)+\Phi_2^2(x)) = |\Phi(x)|^2$.
It is convenient to write the r.h.s. of (20) in terms of $\Pi^{\gtrless}$ and $K^{\gtrless}$. These are defined as [21]

$$\Pi^{>}(x,y) = \Pi(x,y) \qquad \text{for} \qquad x^0 > y^0, \tag{22}$$

$$\Pi^{<}(x,y) = \Pi(x,y) \qquad \text{for} \qquad x^0 < y^0, \tag{23}$$

and equivalently for $K^{\gtrless}$.
Using the relation

$$\begin{aligned} &\int_{\mathcal{C}} dz^0 \int d^3z \quad \Pi(x,z)K(z,x) = \\ &\int_{t^0}^{x^0} dz^0 \int d^3z \, (\Pi^{>}(x,z)K^{<}(z,x) - \Pi^{<}(x,z)K^{>}(z,x)) \end{aligned} \tag{24}$$

finally leads to the e.o.m.

$$\begin{aligned} &(i\partial_{x^0} + \frac{\triangle}{2m} - g[n_c(x) + 2\tilde{n}(x)])\Phi(x) - g\tilde{m}(x)\Phi^*(x) = \\ &2ig\int_{t^0}^{x^0} dz^0 \int d^3z(\Pi^{>}(x,z)K^{<}(z,x) - \Pi^{<}(x,z)K^{>}(z,x)), \end{aligned} \tag{25}$$

together with (21) for $K^{\gtrless}$.
This equation can be seen as a generalization of the time dependent Gross-Pitaevskii equation. In particular, the RHS contains interactions between the condensate and the surrounding non-condensate cloud.

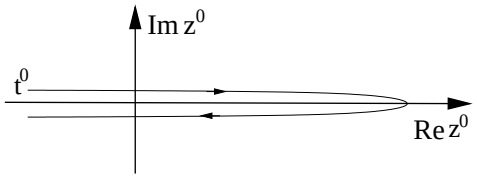

Figure 1: Schwinger-Keldysh contour in the complex z^0-plane

2.1 Derivation of the kinetic equation

Our goal in this section is to derive a kinetic equation for Φ in terms of the non-condensate distribution functions. This equation will have a clear physical interpretation. We have to make some approximations as the very concept of a distribution function $f(\vec{k},\vec{x},t)$ cannot be strictly applied to a quantum mechanical problem. First of all, we will work within the Popov approximation [20], i.e. we neglect the anomalous correlators $<\tilde{\Psi}\tilde{\Psi}>$ and $<\tilde{\Psi}^*\tilde{\Psi}^*>$. In addition, we suppose the coupling constant g to be small as in current experiments the densities near the phase transition are such that the gas is in the weak coupling limit [1].
One then obtains

$$\Pi^{\gtrless}(x,y) \simeq - < \tilde{\Psi}(x)\tilde{\Psi}^*(y) >^{\gtrless} < \tilde{\Psi}^*(x)\tilde{\Psi}(y) >^{\gtrless}, \tag{26}$$

$$K^{\gtrless}(x,y) \simeq g\Phi(x) < \tilde{\Psi}^*(x)\tilde{\Psi}(y) >^{\gtrless}, \tag{27}$$

with $< \tilde{\Psi}(x)\tilde{\Psi}^*(y) >^{<} = < \tilde{\Psi}^*(y)\tilde{\Psi}(x) >$ and so on. We take only the first term of the iterative equation (21) for $K^{\gtrless}$ what leads to a collision term up to order g^2.
Following the well known approach of Kadanoff and Baym [21], we now assume that the correlation length of the system under consideration is sufficiently small and one can write

$$< \tilde{\Psi}^*(y)\tilde{\Psi}(x) > = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{s}-i\omega s^0} f(\vec{p},X), \tag{28}$$

$$< \tilde{\Psi}(x)\tilde{\Psi}^*(y) > = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{s}-i\omega s^0} \left(1+f(\vec{p},X)\right), \tag{29}$$

where $s = x-y$ and $X = \frac{x+y}{2}$ are the relative and center of mass coordinates, respectively, while $p = (\omega,\vec{p})$ denotes the non-condensate quasiparticle four-momentum. The dependence of the correlation functions (28) and (29) on the relative time $s^0 = x^0 - y^0$ is assumed to be determined by these quasiparticles. According to (39), given later, the Hartree-Fock energy is given by $\omega_p = \frac{\vec{p}^2}{2m} + 2g[n_c(x)+\tilde{n}(x)]$ (see also [2]).
Inserting the above approximations in (25) gives

$$\left(i\partial_{x^0} + \frac{\triangle}{2m} - g[n_c(x)+2\tilde{n}(x)]\right)\Phi(x) = 2ig\Phi(x)\int_{t^0}^{x^0} dz^0 \int d^3z$$
$$\times \int \prod_{i=1}^{3}\left(\frac{d^3p_i}{(2\pi)^3}\right) e^{i(m\vec{v}_c-\vec{p}_1-\vec{p}_2+\vec{p}_3)(\vec{z}-\vec{x})-i(\epsilon_c-\omega_1-\omega_2+\omega_3)(z^0-x^0)}$$
$$\times \left[f_2\left(1+f_3\right)f_1 - \left(1+f_2\right)f_3\left(1+f_1\right)\right], \tag{30}$$

where f_i means $f(\vec{p}_i, X)$. In writing (30) the expansion (following [2]) $\Phi(z) \simeq \Phi(x) e^{im\vec{v}_c(\vec{z}-\vec{x})-i\epsilon_c(z^0-x^0)}$ is applied, with $\vec{v}_c$ and ϵ_c being the condensates atoms local velocity and energy, respectively.
In taking the limit $t^0 \longrightarrow -\infty$ one may approximate (again following [2]) the time integral in (30) by

$$\int_{-\infty}^{x^0} dz^0 e^{i(\epsilon_c-\omega_1-\omega_2+\omega_3)(x^0-z^0)} f\left(\frac{x^0+z^0}{2}\right) \simeq f(x^0)\pi\delta(\epsilon_c-\omega_1-\omega_2+\omega_3). \quad (31)$$

The space dependence of the distribution functions may be simplified using a gradient expansion around $\vec{x}$ giving $f(\vec{X}) \simeq f(\vec{x})$. The $\vec{z}$-integral in (30) can then easily be performed and one finally obtains

$$\left(i\partial_{x^0} + \frac{\triangle}{2m} - g[n_c(x) + 2\tilde{n}(x)]\right)\Phi(x) =$$
$$\frac{ig^2\Phi(x)}{(2\pi)^5}\int d^3p_1 d^3p_2 d^3p_3 \quad \delta(m\vec{v}_c - \vec{p}_1 - \vec{p}_2 + \vec{p}_3)\delta(\epsilon_c - \omega_1 - \omega_2 + \omega_3)$$
$$\times\big[f_2(1+f_3)f_1 - (1+f_2)f_3(1+f_1)\big]. \quad (32)$$

This equation is in agreement with the result derived in [2]. The physical interpretation is obvious: the first term in the integrand of the RHS corresponds to the gain term, i.e. it describes the collisions of two non-condensate atoms scattering into a non-condensate and a condensate state. Similarly, the second term describes the inverse process leading to a loss of condensate states. The collision term is shown pictorially in figure 2. Dashed lines correspond to the condensate and full ones to the non-condensate field.

Figure 2: pictorial representation of the collision integral

3 Propagator equation

In the previous section we derived a kinetic equation for the condensate field Φ in terms of the non-condensate distribution function f. In order to obtain

a complete set of equations we thus have to find the Boltzmann equation fulfilled by f. Our starting point is the propagator equation of motion derived within the $\frac{1}{N}$-expansion of the 2PI effective action to NLO (see (36) in [15])

$$-\left(\Box + m^2 + \frac{\lambda}{6N}\left[\Phi_c(x)\Phi_c(x) + G_{cc}(x,x)\right]\right) G_{ab}(x,y) = i\delta_{ab}\delta_{\mathcal{C}}(x-y)$$
$$+\Phi_a(x)K_b(x,y) - i\int_{\mathcal{C}} dz^0 \int d^3z D(x,z)G_{ac}(x,z)G_{cb}(z,y), \tag{33}$$

with (see (34) in [15])

$$D(x,y) = i\frac{\lambda}{3N}\delta_{\mathcal{C}}(x-y) + \frac{\lambda}{3N}K_a(y,x)\Phi_a(x)$$
$$-i\frac{\lambda}{6N}\int_{\mathcal{C}} dz^0 \int d^3z G_{ab}(x,z)G_{ab}(x,z)D(z,y). \tag{34}$$

We now sum equation (33) over suitable values of the indices (a,b), in other words we take $(1,1)+(2,2)+i(1,2)-i(2,1)$. Next we perform the non-relativistic limit and the field redefinition as before, leading to

$$(-i\partial_{x^0} + \frac{\triangle}{2m} - gn(x)) < T\tilde{\Psi}^*(x)\tilde{\Psi}(y) >= \frac{i}{2}\delta_{\mathcal{C}}(x-y) + \Phi^*(x)K(x,y)$$
$$-\frac{i}{2}\int_{\mathcal{C}} dz^0 \int d^3z D(x,z)\big[< T\tilde{\Psi}^*(x)\tilde{\Psi}^*(z) >< T\tilde{\Psi}(z)\tilde{\Psi}(y) >$$
$$+ < T\tilde{\Psi}^*(x)\tilde{\Psi}(z) >< T\tilde{\Psi}^*(z)\tilde{\Psi}(y) >\big], \tag{35}$$

with

$$n(x) = n_c(x) + \tilde{n}(x), \tag{36}$$

$$D(x,y) = 2ig\delta_{\mathcal{C}}(x-y) + 4g\Phi^*(x)K(y,x) + 4g\Phi(x)K^*(y,x)$$
$$+2ig\int_{\mathcal{C}} dz^0 \int d^3z \Pi(x,z)D(z,y), \tag{37}$$

and K given in (21). Note that the correct non-relativistic replacement of the Klein-Gordon operator is here

$$(\Box + m^2) \longrightarrow -2m(-i\partial_{x^0} + \frac{\triangle}{2m}), \tag{38}$$

as it acts on $\tilde{\Psi}^*$.

3.1 Boltzmann equation

One may now derive a Boltzmann equation for the non-condensate distribution function f. Starting from equation (35) we apply again the Popov approximation and we neglect contributions in higher order than g^2, giving

$$
\begin{aligned}
&(-i\partial_{x^0} + \frac{\triangle}{2m} - 2gn(x)) < T\tilde{\Psi}^*(x)\tilde{\Psi}(y) >= \\
&\frac{i}{2}\delta_{\mathcal{C}}(x-y) - 2ig^2 \int_{\mathcal{C}} dz^0 \int d^3z \qquad (39)\\
&\times\big[\Phi^*(x)\Phi(z) < T\tilde{\Psi}^*(x)\tilde{\Psi}(z) >< T\tilde{\Psi}(x)\tilde{\Psi}^*(z) >< T\tilde{\Psi}^*(z)\tilde{\Psi}(y) > \\
&+\Phi^*(z)\Phi(x) < T\tilde{\Psi}(z)\tilde{\Psi}^*(x) >< T\tilde{\Psi}^*(x)\tilde{\Psi}(z) >< T\tilde{\Psi}^*(z)\tilde{\Psi}(y) > \\
&+\Phi^*(x)\Phi(z) < T\tilde{\Psi}^*(z)\tilde{\Psi}(x) >< T\tilde{\Psi}^*(x)\tilde{\Psi}(z) >< T\tilde{\Psi}^*(z)\tilde{\Psi}(y) > \\
&+ < T\tilde{\Psi}^*(x)\tilde{\Psi}(z) >< T\tilde{\Psi}(x)\tilde{\Psi}^*(z) >< T\tilde{\Psi}^*(x)\tilde{\Psi}(z) >< T\tilde{\Psi}^*(z)\tilde{\Psi}(y) >\big].
\end{aligned}
$$

Following, e.g., Blaizot and Iancu (see section 2.3 of [22]) and using the techniques described in the previous section we introduce the relative and center of mass coordinates s and X like before and perform a Fourier transform according to (28) and (29).
As a result the Boltzmann equation finally becomes (with $T = X^0$)

$$
\big[\partial_T + \frac{1}{m}\vec{p}\cdot\partial_{\vec{X}} - 2g(\partial_{\vec{X}} n(X))\cdot\partial_{\vec{p}}\big] f(\vec{p},\vec{X},T) = C_{12} + C_{22}. \qquad (40)
$$

The r.h.s. of the above equation consists of two collision integrals describing two body collisions between non-condensate atoms (C_{22}) and collisions involving one condensate atom (C_{12}). They explicitly read

$$
\begin{aligned}
C_{22} = \frac{2g^2}{(2\pi)^5}\int d^3p_1 d^3p_2 d^3p_3 \delta(\vec{p_1}+\vec{p_2}-\vec{p_3}-\vec{p})\delta(\omega_1+\omega_2-\omega_3-\omega) \\
\times\big[f_1 f_2(1+f_3)(1+f) - f f_3(1+f_2)(1+f_1)\big], \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
C_{12} = \frac{4g^2 n_c(X)}{(2\pi)^2}\int d^3p_1 d^3p_2 \delta(m\vec{v_c}+\vec{p_1}-\vec{p_2}-\vec{p})\delta(\epsilon_c+\omega_1-\omega_2-\omega) \\
\times\big[f_1(1+f_2)(1+f) - f f_2(1+f_1)\big] \\
+\frac{2g^2 n_c(X)}{(2\pi)^2}\int d^3p_1 d^3p_2 \delta(m\vec{v_c}-\vec{p_1}-\vec{p_2}+\vec{p})\delta(\epsilon_c-\omega_1-\omega_2+\omega) \\
\times\big[f_1 f_2(1+f) - f(1+f_2)(1+f_1)\big]. \qquad (42)
\end{aligned}
$$

This result is in agreement with the one derived by Griffin, Nikuni and Zaremba [2] and by Stoof [6].

The approach followed in this paper provides a systematic approximation scheme for the non-equilibrium evolution of a Bose-Einstein condensate. Starting from the nonperturbative equations (25,35) for scalar fields including scattering and memory effects it gives as approximation a practicable system of kinetic equations (32,40) at order g^2 describing the dynamical properties of condensate formation. These equations have already been tested numerically using N-body simulations [23].
A full solution of the coupled 2PI equations (25,35), however, requires extensive numerical work which is beyond the scope of this paper.

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